

Precision heavy flavor contributions to neutral and charged current interactions in DIS

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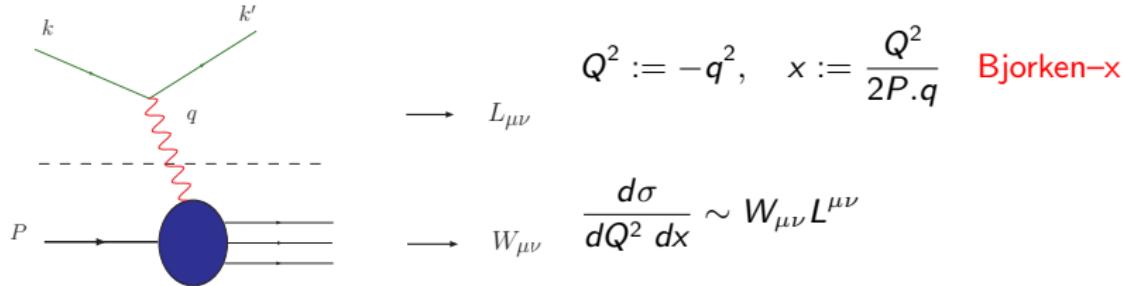
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Introduction

Unpolarized Deep-Inelastic Scattering (DIS):



$$W_{\mu\nu}(q, P, s) = \frac{1}{4\pi} \int d^4\xi \exp(iq\xi) \langle P, s | [J_\mu^{em}(\xi), J_\nu^{em}(0)] | P, s \rangle =$$

$$\frac{1}{2x} \left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) F_L(x, Q^2) + \frac{2x}{Q^2} \left(P_\mu P_\nu + \frac{q_\mu P_\nu + q_\nu P_\mu}{2x} - \frac{Q^2}{4x^2} g_{\mu\nu} \right) F_2(x, Q^2).$$

Structure Functions: $F_{2,L}$

contain light and heavy quark contributions.

Introduction

What are the Key-Goals ?

- ▶ precision measurement of the pdfs [unpolarized and polarized]
- ▶ precision determination of $\alpha_s(M_Z)$, m_c and possibly m_b [ABM collaboration; ABMP 2016 upcoming]
- ▶ determination of higher twist terms (non-singlet, singlet; unpolarized, polarized) [JB, Böttcher: 2008,2012]
- ▶ test of important sum rules (unpolarized, polarized: twist 2 & 3)

The EIC, running as a high luminosity facility, could greatly contribute to all these subjects.

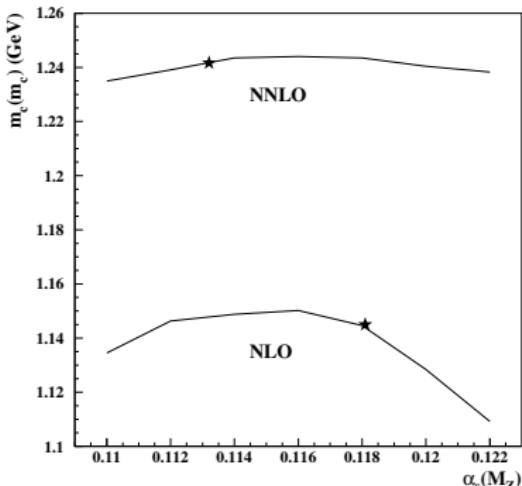
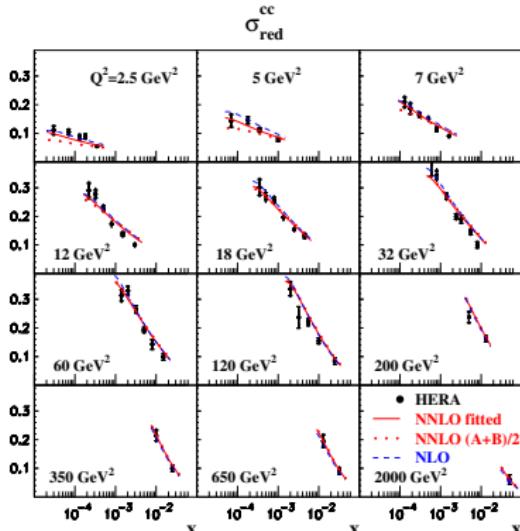
$\alpha_s(M_Z^2)$ from NNLO DIS(+) analyses

	$\alpha_s(M_Z^2)$	
BBG	0.1134 $^{+0.0019}_{-0.0021}$	valence analysis, NNLO
GRS	0.112	valence analysis, NNLO
ABKM	0.1135 ± 0.0014	HQ: FFNS $N_f = 3$
JR	0.1128 ± 0.0010	dynamical approach
JR	0.1162 ± 0.0006	including NLO-jets
MSTW	0.1171 ± 0.0014	
Thorne	0.1136	[DIS+DY+HT*] (2014)
ABM11 _J	$0.1134 - 0.1149 \pm 0.0012$	Tevatron jets (NLO) incl.
ABM13	0.1133 ± 0.0011	
ABM13	0.1132 ± 0.0011	(without jets)
ABM16	0.1149 ± 0.0009	+ new HERA, + $t\bar{t}$
CTEQ	0.1159..0.1162	
CTEQ	0.1140	(without jets)
NN21	$0.1174 \pm 0.0006 \pm 0.0001$	
Gehrman et al.	$0.1131 ^{+0.0028}_{-0.0022}$	e^+e^- thrust
Abbate et al.	0.1140 ± 0.0015	e^+e^- thrust
BBG	$0.1141 ^{+0.0020}_{-0.0022}$	valence analysis, N³LO

$$\Delta_{\text{TH}}\alpha_s = \alpha_s(\text{N}^3\text{LO}) - \alpha_s(\text{NNLO}) + \Delta_{\text{HQ}} = +0.0009 \pm 0.0006_{\text{HQ}}$$

NNLO accuracy is needed to analyze the world data. \implies NNLO HQ corrections needed.

Deep-Inelastic Scattering (DIS):



NNLO:

S. Alekhin, J. Blümlein, K. Daum, K. Lipka, Phys.Lett. B720 (2013) 172
[1212.2355]

$$m_c(m_c) = 1.252 \pm 0.02(\text{exp}) \quad {}^{+0.03}_{-0.02} \quad (\text{scale}) \quad {}^{+0.00}_{-0.07} \quad (\text{thy}) \text{GeV},$$

$$m_b(m_b) = 3.83 \pm 0.12 \text{ GeV}$$

$$m_t(m_t) = 160.9 \pm 1.1 \text{ GeV}$$

Yet approximate NNLO treatment [Kawamura et al. [1205.5227]].

The massless DIS cross sections

- ▶ 3-loop evolution: Moch, Vermaseren, Vogt: 2004; a series of contributions independently confirmed by JB, Schneider et al. 2010,2014,2016 (unpolarized)
- ▶ 3-loop evolution: Moch, Vermaseren, Vogt: 2014 (polarized).
- ▶ 3-loop photon exchange Wilson coefficient: Moch, Vermaseren, Vogt: 2005
- ▶ 3-loop CC Wilson coefficients: Moch, Vermaseren, Vogt (et al.): 2008, 2016

NNLO Heavy Flavor Corrections to DIS

Factorization of the Structure Functions

At leading twist the structure functions factorize in terms of a Mellin convolution

$$F_{(2,L)}(x, Q^2) = \sum_j \underbrace{\mathbb{C}_{j,(2,L)} \left(x, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2} \right)}_{\text{perturbative}} \otimes \underbrace{f_j(x, \mu^2)}_{\text{nonpert.}}$$

into (pert.) **Wilson coefficients** and (nonpert.) **parton distribution functions (PDFs)**.

\otimes denotes the Mellin convolution

$$f(x) \otimes g(x) \equiv \int_0^1 dy \int_0^1 dz \delta(x - yz) f(y) g(z) .$$

The subsequent calculations are performed in Mellin space, where \otimes reduces to a multiplication, due to the Mellin transformation

$$\hat{f}(N) = \int_0^1 dx x^{N-1} f(x) .$$

Wilson coefficients:

$$\mathbb{C}_{j,(2,L)}\left(N, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right) = \textcolor{blue}{C}_{j,(2,L)}\left(N, \frac{Q^2}{\mu^2}\right) + H_{j,(2,L)}\left(N, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right).$$

At $Q^2 \gg m^2$ the heavy flavor part

$$H_{j,(2,L)}\left(N, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right) = \sum_i \textcolor{blue}{C}_{i,(2,L)}\left(N, \frac{Q^2}{\mu^2}\right) \textcolor{red}{A}_{ij}\left(\frac{m^2}{\mu^2}, N\right)$$

[Buza, Matiounine, Smith, van Neerven 1996 Nucl.Phys.B] factorizes into the light flavor Wilson coefficients $\textcolor{blue}{C}$ and the massive operator matrix elements (OMEs) of local operators O_i between partonic states j

$$\textcolor{red}{A}_{ij}\left(\frac{m^2}{\mu^2}, N\right) = \langle j | O_i | j \rangle.$$

→ additional Feynman rules with local operator insertions for partonic matrix elements.

The unpolarized light flavor Wilson coefficients are known up to NNLO

[Moch, Vermaseren, Vogt, 2005 Nucl.Phys.B].

For $F_2(x, Q^2)$: at $Q^2 \gtrsim 10m^2$ the asymptotic representation holds at the 1% level.

The Wilson Coefficients at large Q^2

2014 $L_{q,(2,L)}^{\text{NS}}(N_F + 1) = a_s^2 \left[A_{qq,Q}^{(2),\text{NS}}(N_F + 1) \delta_2 + \hat{C}_{q,(2,L)}^{(2),\text{NS}}(N_F) \right]$
 $+ a_s^3 \left[A_{qq,Q}^{(3),\text{NS}}(N_F + 1) \delta_2 + A_{qg,Q}^{(2),\text{NS}}(N_F + 1) C_{q,(2,L)}^{(1),\text{NS}}(N_F + 1) + \hat{C}_{q,(2,L)}^{(3),\text{NS}}(N_F) \right]$

2010 $L_{q,(2,L)}^{\text{PS}}(N_F + 1) = a_s^3 \left[A_{qq,Q}^{(3),\text{PS}}(N_F + 1) \delta_2 + A_{gg,Q}^{(2)}(N_F) N_F \tilde{C}_{g,(2,L)}^{(1)}(N_F + 1) + N_F \hat{\tilde{C}}_{g,(2,L)}^{(3),\text{PS}}(N_F)$

2010 $L_{g,(2,L)}^{\text{S}}(N_F + 1) = a_s^2 A_{gg,Q}^{(1)}(N_F + 1) N_F \tilde{C}_{g,(2,L)}^{(1)}(N_F + 1) + a_s^3 \left[A_{qg,Q}^{(3)}(N_F + 1) \delta_2 \right.$
 $+ A_{gg,Q}^{(1)}(N_F + 1) N_F \tilde{C}_{g,(2,L)}^{(2)}(N_F + 1) + A_{gg,Q}^{(2)}(N_F + 1) N_F \tilde{C}_{g,(2,L)}^{(1)}(N_F + 1)$
 $\left. + A_{Qg}^{(1)}(N_F + 1) N_F \tilde{C}_{q,(2,L)}^{(2),\text{PS}}(N_F + 1) + N_F \hat{\tilde{C}}_{g,(2,L)}^{(3)}(N_F) \right],$

2014 $H_{q,(2,L)}^{\text{PS}}(N_F + 1) = a_s^2 \left[A_{Qq}^{(2),\text{PS}}(N_F + 1) \delta_2 + \tilde{C}_{q,(2,L)}^{(2),\text{PS}}(N_F + 1) \right] + a_s^3 \left[A_{Qq}^{(3),\text{PS}}(N_F + 1) \delta_2 \right.$
 $+ \tilde{C}_{q,(2,L)}^{(3),\text{PS}}(N_F + 1) + A_{gg,Q}^{(2)}(N_F + 1) \tilde{C}_{g,(2,L)}^{(1)}(N_F + 1)$
 $\left. + A_{Qq}^{(2),\text{PS}}(N_F + 1) C_{q,(2,L)}^{(1),\text{NS}}(N_F + 1) \right],$

$H_{g,(2,L)}^{\text{S}}(N_F + 1) = a_s \left[A_{Qg}^{(1)}(N_F + 1) \delta_2 + \tilde{C}_{g,(2,L)}^{(1)}(N_F + 1) \right] + a_s^2 \left[A_{Qg}^{(2)}(N_F + 1) \delta_2 \right.$
 $+ A_{Qg}^{(1)}(N_F + 1) C_{q,(2,L)}^{(1),\text{NS}}(N_F + 1) + A_{gg,Q}^{(1)}(N_F + 1) \tilde{C}_{g,(2,L)}^{(1)}(N_F + 1)$
 $\left. + \tilde{C}_{g,(2,L)}^{(2)}(N_F + 1) \right] + a_s^3 \left[A_{Qg}^{(3)}(N_F + 1) \delta_2 + A_{Qg}^{(2)}(N_F + 1) C_{q,(2,L)}^{(1),\text{NS}}(N_F + 1) \right.$
 $+ A_{gg,Q}^{(2)}(N_F + 1) \tilde{C}_{g,(2,L)}^{(1)}(N_F + 1) + A_{Qg}^{(1)}(N_F + 1) \left\{ C_{q,(2,L)}^{(2),\text{NS}}(N_F + 1) \right.$
 $\left. + \tilde{C}_{q,(2,L)}^{(2),\text{PS}}(N_F + 1) \right\} + A_{gg,Q}^{(1)}(N_F + 1) \tilde{C}_{g,(2,L)}^{(2)}(N_F + 1) + \tilde{C}_{g,(2,L)}^{(3)}(N_F + 1)$

[Ablinger et al. 2010, Ablinger et al., 2014a, Ablinger et al., 2014b]

Variable Flavor Number Scheme

$$\begin{aligned} f_k(n_f + 1, \mu^2) + f_{\bar{k}}(n_f + 1, \mu^2) &= A_{qq,Q}^{\text{NS}}\left(n_f, \frac{\mu^2}{m^2}\right) \otimes \left[f_k(n_f, \mu^2) + f_{\bar{k}}(n_f, \mu^2) \right] \\ &\quad + \tilde{A}_{qq,Q}^{\text{PS}}\left(n_f, \frac{\mu^2}{m^2}\right) \otimes \Sigma(n_f, \mu^2) + \tilde{A}_{qg,Q}^{\text{S}}\left(n_f, \frac{\mu^2}{m^2}\right) \otimes G(n_f, \mu^2) \\ f_{Q+\bar{Q}}(n_f + 1, \mu^2) &= \tilde{A}_{Qq}^{\text{PS}}\left(n_f, \frac{\mu^2}{m^2}\right) \otimes \Sigma(n_f, \mu^2) + \tilde{A}_{Qg}^{\text{S}}\left(n_f, \frac{\mu^2}{m^2}\right) \otimes G(n_f, \mu^2) . \\ G(n_f + 1, \mu^2) &= A_{gq,Q}^{\text{S}}\left(n_f, \frac{\mu^2}{m^2}\right) \otimes \Sigma(n_f, \mu^2) + A_{gg,Q}^{\text{S}}\left(n_f, \frac{\mu^2}{m^2}\right) \otimes G(n_f, \mu^2) . \\ \Sigma(n_f + 1, \mu^2) &= \sum_{k=1}^{n_f+1} \left[f_k(n_f + 1, \mu^2) + f_{\bar{k}}(n_f + 1, \mu^2) \right] \\ &= \left[A_{qq,Q}^{\text{NS}}\left(n_f, \frac{\mu^2}{m^2}\right) + n_f \tilde{A}_{qq,Q}^{\text{PS}}\left(n_f, \frac{\mu^2}{m^2}\right) + \tilde{A}_{Qq}^{\text{PS}}\left(n_f, \frac{\mu^2}{m^2}\right) \right] \\ &\quad \otimes \Sigma(n_f, \mu^2) \\ &\quad + \left[n_f \tilde{A}_{qg,Q}^{\text{S}}\left(n_f, \frac{\mu^2}{m^2}\right) + \tilde{A}_{Qg}^{\text{S}}\left(n_f, \frac{\mu^2}{m^2}\right) \right] \otimes G(n_f, \mu^2) \end{aligned}$$

All master integrals for $A_{gg}^{(3)}$ have been completed (June 2015).

Calculation of the 3-loop operator matrix elements

The OMEs are calculated using the QCD Feynman rules together with the following operator insertion Feynman rules:

$$\begin{aligned}
 & \text{Diagram 1: } p_i \xrightarrow{\otimes} p_j \\
 & \delta^{ij} \Delta \gamma_{\pm} (\Delta \cdot p)^{N-1}, \quad N \geq 1
 \end{aligned}$$

$$\begin{aligned}
 & \text{Diagram 2: } p_1, i \xrightarrow{\otimes} p_2, j \\
 & g t_{ji}^a \Delta^\mu \Delta \gamma_{\pm} \sum_{j=0}^{N-2} (\Delta \cdot p_1)^j (\Delta \cdot p_2)^{N-j-2}, \quad N \geq 2
 \end{aligned}$$

$$\begin{aligned}
 & \text{Diagram 3: } p_1, i \xrightarrow{\otimes} p_2, j \\
 & \text{with internal loop } \mu, a
 \end{aligned}$$

$$\begin{aligned}
 & \text{Diagram 4: } p_1, i \xrightarrow{\otimes} p_2, j \\
 & \text{with internal loops } p_3, \mu, a \text{ and } p_4, \nu, b
 \end{aligned}$$

$$\begin{aligned}
 & \text{Diagram 5: } p_1, i \xrightarrow{\otimes} p_2, j \\
 & \text{with internal loops } p_3, \mu, a, p_4, \nu, b \text{ and } p_5, \rho, c
 \end{aligned}$$

$$\begin{aligned}
 & g^2 \Delta^\mu \Delta^\nu \Delta \gamma_{\pm} \sum_{j=0}^{N-3} \sum_{l=j+1}^{N-2} (\Delta p_2)^j (\Delta p_1)^{N-l-2} \\
 & [(t^a t^b)_{ji} (\Delta p_1 + \Delta p_4)^{l-j-1} + (t^b t^a)_{ji} (\Delta p_1 + \Delta p_3)^{l-j-1}], \quad N \geq 3
 \end{aligned}$$

$$\begin{aligned}
 & g^3 \Delta_\mu \Delta_\nu \Delta_\rho \Delta \gamma_{\pm} \sum_{j=0}^{N-4} \sum_{l=j+1}^{N-3} \sum_{m=l+1}^{N-2} (\Delta p_2)^j (\Delta p_1)^{N-m-2} \\
 & [(t^a t^b t^c)_{ji} (\Delta p_4 + \Delta p_5 + \Delta p_1)^{l-j-1} (\Delta p_5 + \Delta p_1)^{m-l-1} \\
 & + (t^a t^b t^c)_{ji} (\Delta p_4 + \Delta p_5 + \Delta p_1)^{l-j-1} (\Delta p_4 + \Delta p_1)^{m-l-1} \\
 & + (t^b t^c t^a)_{ji} (\Delta p_3 + \Delta p_5 + \Delta p_1)^{l-j-1} (\Delta p_5 + \Delta p_1)^{m-l-1} \\
 & + (t^b t^c t^a)_{ji} (\Delta p_3 + \Delta p_5 + \Delta p_1)^{l-j-1} (\Delta p_3 + \Delta p_1)^{m-l-1} \\
 & + (t^c t^a t^b)_{ji} (\Delta p_3 + \Delta p_4 + \Delta p_1)^{l-j-1} (\Delta p_4 + \Delta p_1)^{m-l-1} \\
 & + (t^c t^a t^b)_{ji} (\Delta p_3 + \Delta p_4 + \Delta p_1)^{l-j-1} (\Delta p_3 + \Delta p_1)^{m-l-1}], \quad N \geq 4
 \end{aligned}$$

$$\gamma_+ = 1, \quad \gamma_- = \gamma_5.$$

$$\begin{aligned}
 & \text{Feynman rule: } p, \nu, b \xrightarrow{\otimes} p, \mu, a \\
 & \frac{1+(-1)^N}{2} \delta^{ab} (\Delta \cdot p)^{N-2} \\
 & [g_{\mu\nu} (\Delta \cdot p)^2 - (\Delta_\mu p_\nu + \Delta_\nu p_\mu) \Delta \cdot p + p^2 \Delta_\mu \Delta_\nu], \quad N \geq 2
 \end{aligned}$$

$$\begin{aligned}
 & \text{Feynman rule: } p_1, \mu, a \xrightarrow{\otimes} p_3, \lambda, c \\
 & \text{with internal loop } p_2, \nu, b \\
 & -ig \frac{1+(-1)^N}{2} f^{abc} \left(\right. \\
 & \left[(\Delta_\nu g_{\mu\lambda} - \Delta_\lambda g_{\mu\nu}) \Delta \cdot p_1 + \Delta_\mu (p_{1,\nu} \Delta_\lambda - p_{1,\lambda} \Delta_\nu) \right] (\Delta \cdot p_1)^{N-2} \\
 & + \Delta_\lambda \left[\Delta \cdot p_1 p_{2,\mu} \Delta_\nu + \Delta \cdot p_2 p_{1,\nu} \Delta_\mu - \Delta \cdot p_1 \Delta \cdot p_2 g_{\mu\nu} - p_1 \cdot p_2 \Delta_\mu \Delta_\nu \right] \\
 & \times \sum_{j=0}^{N-3} (-\Delta \cdot p_1)^j (\Delta \cdot p_2)^{N-3-j} \\
 & \left. + \left\{ \begin{array}{l} p_1 \rightarrow p_2 \rightarrow p_3 \rightarrow p_1 \\ \mu \rightarrow \nu \rightarrow \lambda \rightarrow \mu \end{array} \right\} + \left\{ \begin{array}{l} p_1 \rightarrow p_3 \rightarrow p_2 \rightarrow p_1 \\ \mu \rightarrow \lambda \rightarrow \nu \rightarrow \mu \end{array} \right\} \right), \quad N \geq 2
 \end{aligned}$$

$$\begin{aligned}
 & \text{Feynman rule: } p_1, \mu, a \xrightarrow{\otimes} p_4, \sigma, d \\
 & \text{with internal loops } p_2, \nu, b \text{ and } p_3, \lambda, c \\
 & g^2 \frac{1+(-1)^N}{2} \left(f^{abc} f^{cd\epsilon} O_{\mu\nu\lambda\sigma}(p_1, p_2, p_3, p_4) \right. \\
 & + f^{a\epsilon c} f^{bde} O_{\mu\lambda\sigma\tau}(p_1, p_3, p_2, p_4) + f^{ade} f^{bce} O_{\mu\nu\tau\lambda}(p_1, p_4, p_2, p_3) \left. \right), \\
 & O_{\mu\nu\lambda\sigma}(p_1, p_2, p_3, p_4) = \Delta_\nu \Delta_\lambda \left\{ -g_{\mu\sigma} (\Delta \cdot p_3 + \Delta \cdot p_4)^{N-2} \right. \\
 & + [p_{4,\mu} \Delta_\sigma - \Delta \cdot p_4 g_{\mu\sigma}] \sum_{i=0}^{N-3} (\Delta \cdot p_3 + \Delta \cdot p_4)^i (\Delta \cdot p_4)^{N-3-i} \\
 & - [p_{1,\sigma} \Delta_\mu - \Delta \cdot p_1 g_{\mu\sigma}] \sum_{i=0}^{N-3} (-\Delta \cdot p_1)^i (\Delta \cdot p_3 + \Delta \cdot p_4)^{N-3-i} \\
 & + [\Delta \cdot p_1 \Delta \cdot p_4 g_{\mu\sigma} + p_1 \cdot p_4 \Delta_\mu \Delta_\sigma - \Delta \cdot p_4 p_{1,\sigma} \Delta_\mu - \Delta \cdot p_1 p_{4,\mu} \Delta_\sigma] \\
 & \times \sum_{i=0}^{N-4} \sum_{j=0}^i (-\Delta \cdot p_1)^{N-4-i} (\Delta \cdot p_3 + \Delta \cdot p_4)^{i-j} (\Delta \cdot p_4)^j \\
 & \left. - \left\{ \begin{array}{l} p_1 \leftrightarrow p_2 \\ \mu \leftrightarrow \nu \end{array} \right\} - \left\{ \begin{array}{l} p_3 \leftrightarrow p_4 \\ \lambda \leftrightarrow \sigma \end{array} \right\} + \left\{ \begin{array}{l} p_1 \leftrightarrow p_2, p_3 \leftrightarrow p_4 \\ \mu \leftrightarrow \nu, \lambda \leftrightarrow \sigma \end{array} \right\} \right), \quad N \geq 2
 \end{aligned}$$

Integration by parts

We use **Reduze** [A. von Manteuffel, C. Studerus, 2012] to express all scalar integrals required in the calculation in terms of a small(er) set of master integrals.

Reduze is a **C++** program based on **Laporta's algorithm**.

$$(\Delta \cdot k)^N \rightarrow \sum_{N=0}^{\infty} x^N (\Delta \cdot k)^N = \frac{1}{1 - x\Delta \cdot k}$$

⇒ additional propagator.

Number of master integrals:

$A_{qq,Q}^{(3),NS}$ → 35 master integrals ✓.

$A_{gq,Q}^{(3)}$ → 41 master integrals ✓.

$A_{Qq}^{(3),PS}$ → 66 master integrals ✓.

$A_{gg,Q}^{(3)}$ → 205 master integrals ✓.

$A_{Qg}^{(3)}$ → 340 master integrals. (224 done by June 2015.)

116 master integrals to be done ⇒ CIS-type

24 integral families are required and implemented in Reduze.

Calculation of the master integrals

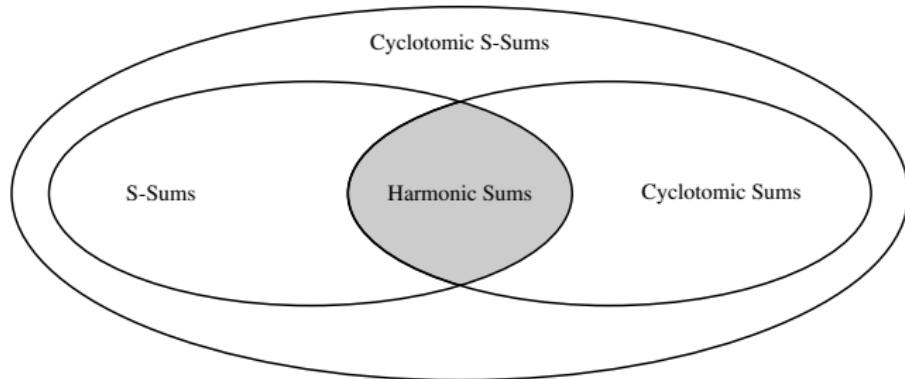
For the calculation of the master integrals we use a wide variety of tools:

- ▶ Hypergeometric functions.
- ▶ Summation methods based on difference fields, implemented in the Mathematica program **Sigma** [C. Schneider, 2005–].
 - ▶ Reduction of the sums to a small number of key sums.
 - ▶ Expansion the summands in ε .
 - ▶ Simplification by symbolic summation algorithms based on $\Pi\Sigma$ -fields [Karr 1981 J. ACM, Schneider 2005–].
 - ▶ Harmonic sums, polylogarithms and their various generalizations are algebraically reduced using the package **HarmonicSums** [Ablinger 2010, 2013, Ablinger, Blümlein, Schneider 2011, 2013].
- ▶ Mellin-Barnes representations.
- ▶ In the case of **convergent** massive 3-loop Feynman integrals, they can be performed in terms of **Hyperlogarithms** [Generalization of a method by F. Brown, 2008, to non-vanishing masses and local operators].
- ▶ Systems of Differential Equations.
- ▶ Almkvist-Zeilberger Theorem as Integration Method.

Spill-Off: New Mathematical Function Classes and Algebras

- ▶ 1998: Harmonic Sums [Vermaseren; JB]
- ▶ 1999: Harmonic Polylogarithms [Remiddi, Vermaseren]
- ▶ 2001: Generalized Harmonic Sums [Moch, Uwer, Weinzierl]
- ▶ 2004: Infinite harmonic (inverse) binomial sums [Davydychev, Kalmykov; Weinzierl]
- ▶ 2011: (generalized) Cyclotomic Harmonic Sums, polylogarithms and numbers [Ablinger, JB, Schneider]
- ▶ 2013: Systematic Theory of Generalized Harmonic Sums, polylogarithms and numbers [Ablinger, JB, Schneider]
- ▶ 2014: Finite nested Generalized Cyclotomic Harmonic Sums with (inverse) Binomial Weights [Ablinger, JB, Raab, Schneider]
- ▶ 2016: Elliptic integrals with (involved) rational arguments appear in part of the functions of our project already as base cases. They stem from Heun equations. [since April 2016.] [Ablinger, Behring, JB, De Freitas, van Hoeij, Imamoglu, Raab, Schneider, DESY16-147].

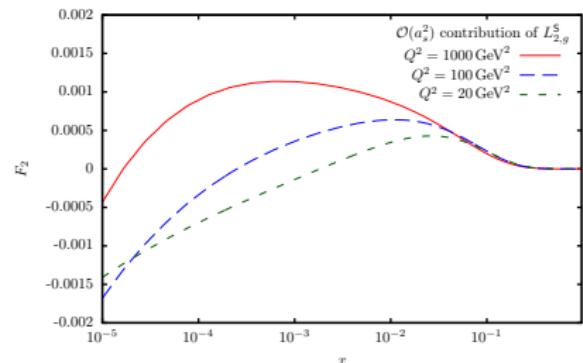
Particle Physics Generates **NEW** Mathematics.



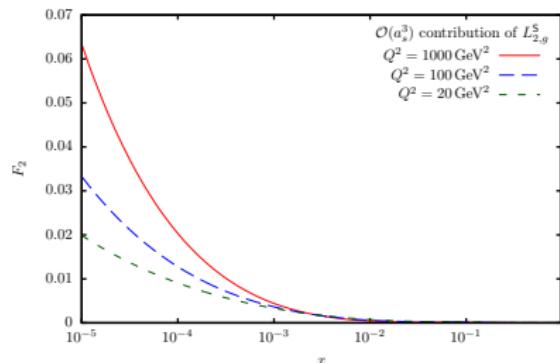
Nested (inverse) binomial sums; CIS functions [Elliptic Integrals and iterations on them]
.....

More and more onion skins to be added during these calculations.

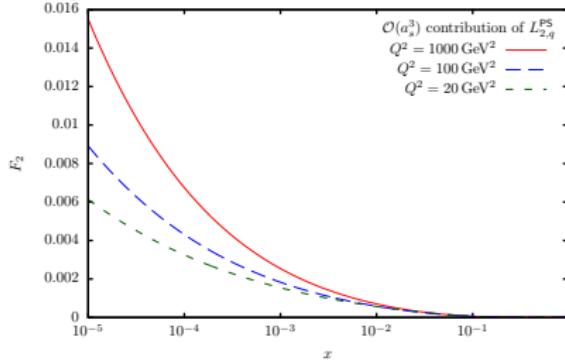
Numerical Results : $L_{g,2}^S$ and $L_{q,2}^{PS}$



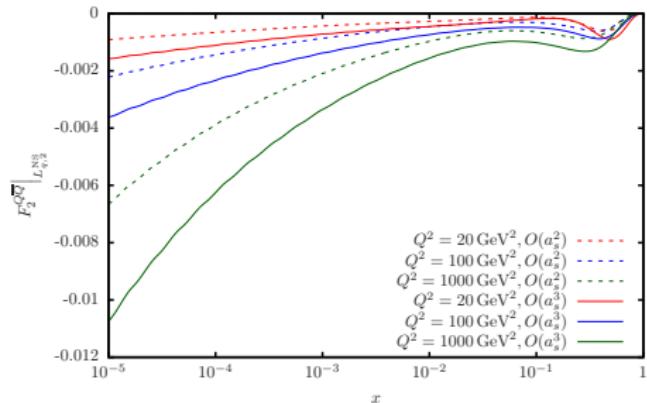
$$O(a_s^2) \quad L_{2,g}^S$$



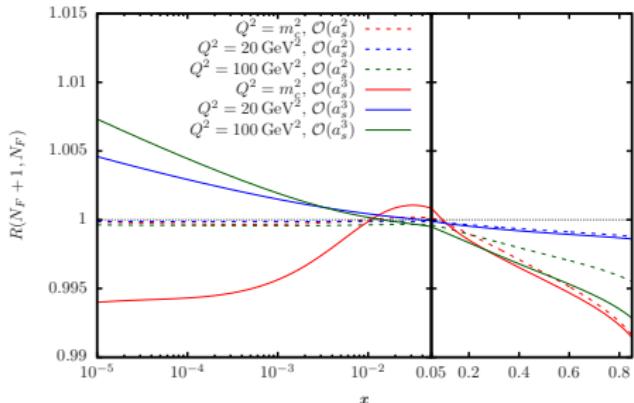
$$O(a_s^3) \quad L_{2,g}^S$$



$$L_{q,2}^{\text{PS}}$$

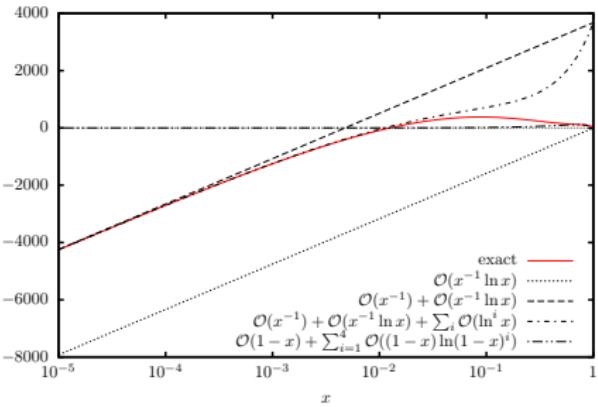


Contribution to $F_2(x, Q^2)$

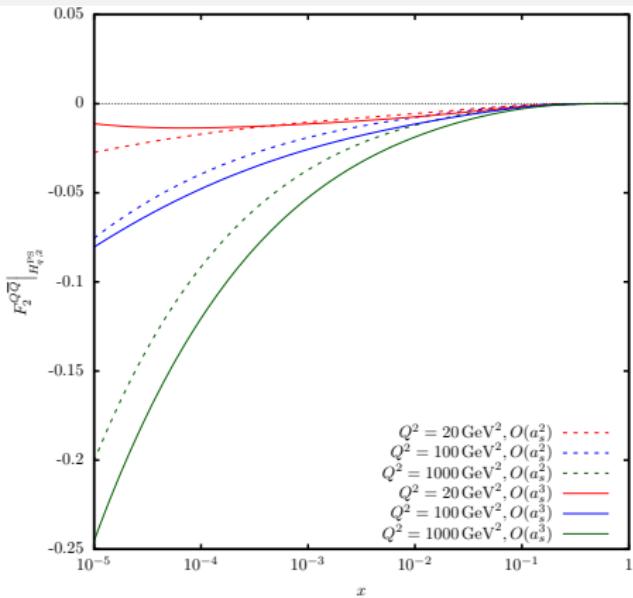


VFNS matching

$$H_{q,2}^{\text{PS}}$$



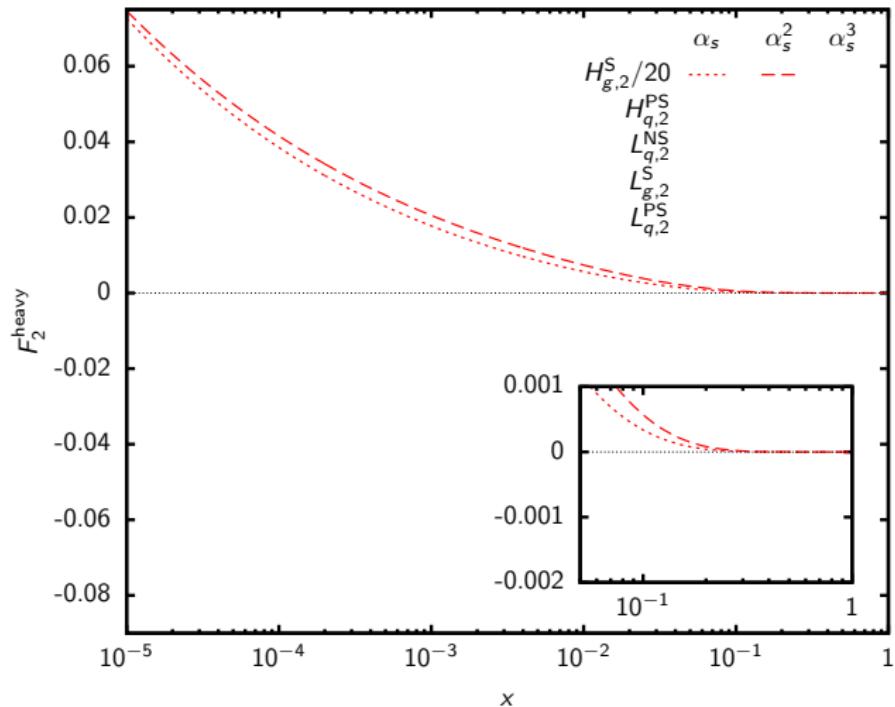
$$a_{Qq}^{(3),\text{PS}}$$



Contribution to $F_2(x, Q^2)$

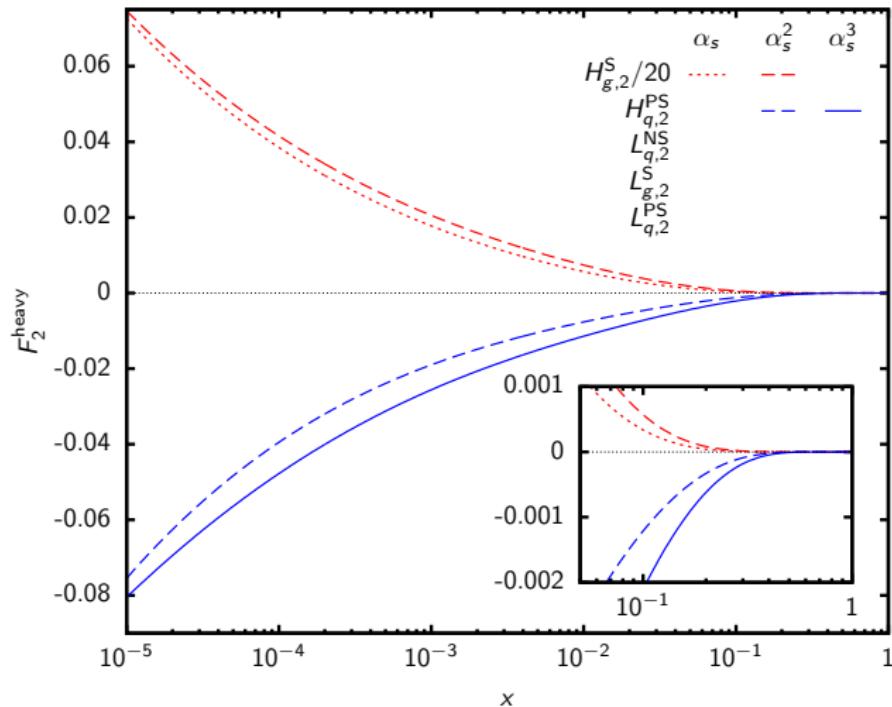
The leading small x approximation corresponding to CCH, 1991, departs from the physical result everywhere except for $x = 1$.

The present NC corrections to $F_2(x, Q^2)$



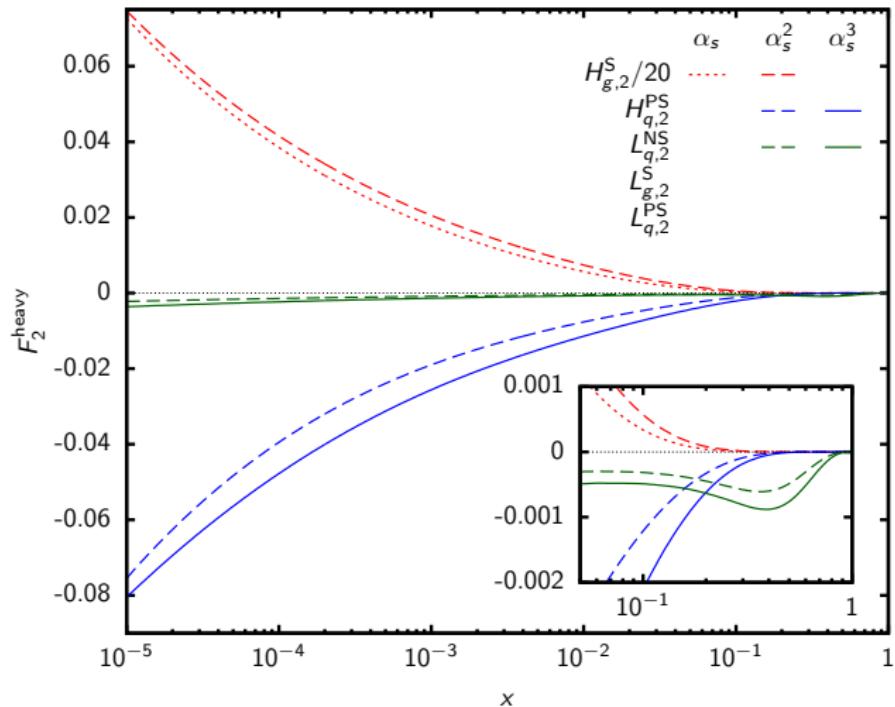
$Q^2 = 100 \text{ GeV}^2$ [$H_{g,2}^S$ scaled down by a factor 20.]

The present NC corrections to $F_2(x, Q^2)$



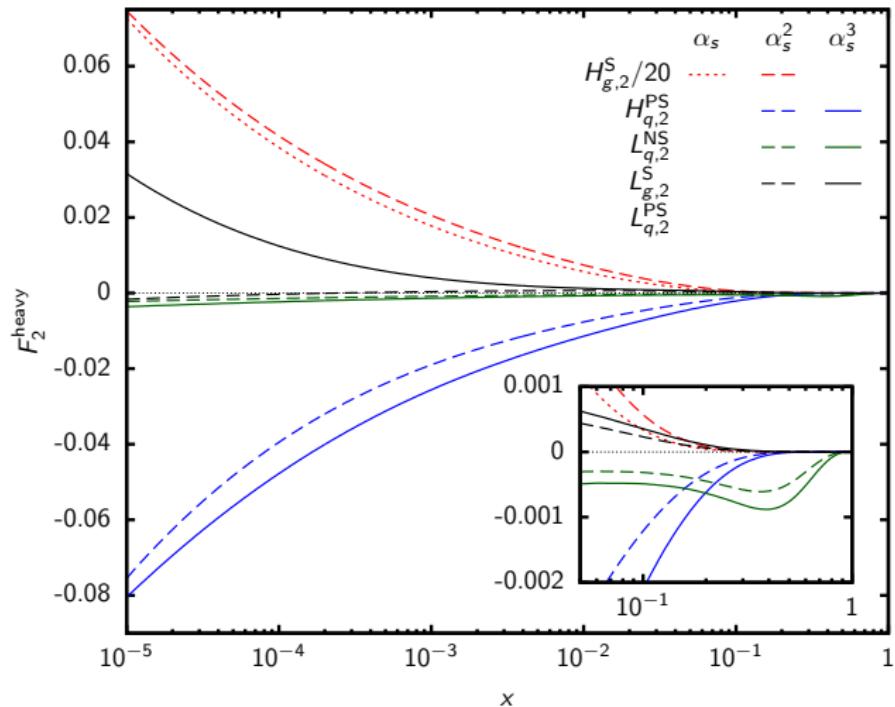
$Q^2 = 100 \text{ GeV}^2$ [$H_{g,2}^S$ scaled down by a factor 20.]

The present NC corrections to $F_2(x, Q^2)$



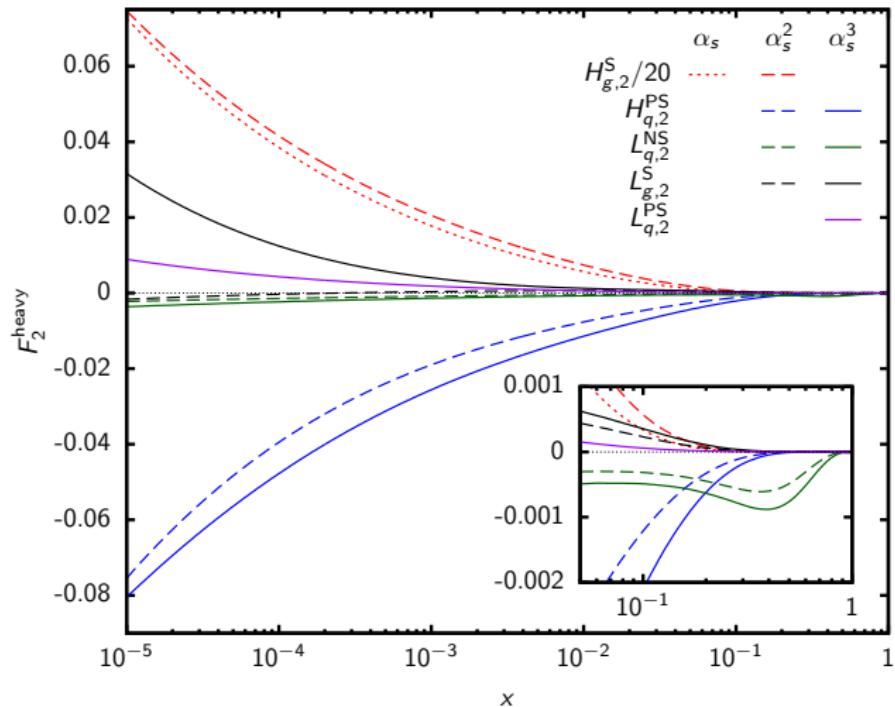
$Q^2 = 100 \text{ GeV}^2$ [$H_{g,2}^S$ scaled down by a factor 20.]

The present NC corrections to $F_2(x, Q^2)$



$Q^2 = 100 \text{ GeV}^2$ [$H_{g,2}^S$ scaled down by a factor 20.]

The present NC corrections to $F_2(x, Q^2)$



$Q^2 = 100 \text{ GeV}^2$ [$H_{g,2}^S$ scaled down by a factor 20.]

3-Loop OME: $A_{gg,Q}$

$$\begin{aligned}
a_{gg,Q}^{(3)} = & \frac{1 + (-1)^N}{2} \left\{ c_F^2 T_F \left[\frac{16(N^2 + N + 2)}{N^2(N+1)^2} \sum_{i=1}^N \frac{\binom{2i}{i} \left(\sum_{j=1}^i \frac{4^j S_1(j-1)}{\binom{2j}{j} j^2} - 7\zeta_3 \right)}{4^i (i+1)^2} - \frac{4P_{69} S_1^2}{3(N-1)N^4(N+1)^4(N+2)} \right. \right. \\
& + \tilde{\gamma}_{gq}^{(0)} \left(\frac{128(S_{-4} - S_{-3}S_1 + S_{-3,1} + 2S_{-2,2})}{3N(N+1)(N+2)} + \frac{4(5N^2 + 5N - 22)S_1^2 S_2}{3N(N+1)(N+2)} + \dots \right) + \dots \Big] \\
& + c_A c_F T_F \left[\frac{16P_{42}}{3(N-1)N^2(N+1)^2(N+2)} \sum_{i=1}^N \frac{\binom{2i}{i} \left(\sum_{j=1}^i \frac{4^j S_1(j-1)}{\binom{2j}{j} j^2} - 7\zeta_3 \right)}{4^i (i+1)^2} + \frac{32P_2 S_{-2,2}}{(N-1)N^2(N+1)^2(N+2)} \right. \\
& - \frac{64P_{14} S_{-2,1,1}}{3(N-1)N^2(N+1)^2(N+2)} - \frac{16P_{23} S_{-4}}{3(N-1)N^2(N+1)^2(N+2)} + \frac{4P_{63} S_4}{3(N-2)(N-1)N^2(N+1)^2(N+2)} + \dots \Big] \\
& + c_A^2 T_F \left[- \frac{4P_{46}}{3(N-1)N^2(N+1)^2(N+2)} \sum_{i=1}^N \frac{\binom{2i}{i} \left(\sum_{j=1}^i \frac{4^j S_1(j-1)}{\binom{2j}{j} j^2} - 7\zeta_3 \right)}{4^i (i+1)^2} + \frac{256P_5 S_{-2,2}}{9(N-1)N^2(N+1)^2(N+2)} \right. \\
& + \frac{32P_{30} S_{-2,1,1} + 16P_{35} S_{-3,1} + 16P_{44} S_{-4}}{9(N-1)N^2(N+1)^2(N+2)} + \frac{16P_{52} S_{-2}^2}{27(N-1)N^2(N+1)^2(N+2)} + \frac{8P_{36} S_2^2}{9(N-1)N^2(N+1)^2} + \dots \Big] \\
& + c_F T_F^2 \left[- \frac{16P_{48} \binom{2N}{N} 4^{-N} \left(\sum_{i=1}^N \frac{4^i S_1(i-1)}{\binom{2i}{i} i^2} - 7\zeta_3 \right)}{3(N-1)N(N+1)^2(N+2)(2N-3)(2N-1)} - \frac{32P_{86} S_1}{81(N-1)N^4(N+1)^4(N+2)(2N-3)(2N-1)} \right. \\
& + \frac{16P_{45} S_1^2}{27(N-1)N^3(N+1)^3(N+2)} - \frac{16P_{45} S_2}{9(N-1)N^3(N+1)^3(N+2)} + \dots \Big] + \dots \Big\} \quad (1)
\end{aligned}$$

Also, with this calculation we were able to re-derive the three loop anomalous dimension $\gamma_{gg}^{(3)}$ for the terms $\propto T_F$, and obtained

agreement with the literature.

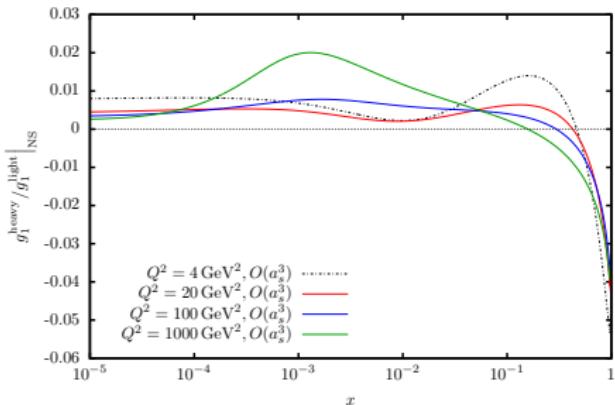
Moments for graphs with two massive lines ($m_1 \neq m_2$)

$$\begin{aligned}
a_{Qg}^{(3)}(N=6) = & \frac{1}{2} \left\{ T_F^2 C_A \left\{ \frac{69882273800453}{367569090000} - \frac{395296}{19845} \zeta_3 + \frac{1316809}{39690} \zeta_2 + \frac{832369820129}{14586075000} x + \frac{1511074426112}{624023544375} x^2 - \frac{84840004938801319}{690973782403905000} x^3 \right. \right. \\
& + \ln\left(\frac{m_2^2}{\mu^2}\right) \left[\frac{11771644229}{194481000} + \frac{78496}{2205} \zeta_2 - \frac{1406143531}{69457500} x - \frac{105157957}{180093375} x^2 + \frac{2287164970759}{7669816654500} x^3 \right] \\
& + \ln^2\left(\frac{m_2^2}{\mu^2}\right) \left[\frac{2668087}{79380} + \frac{112669}{661500} x - \frac{49373}{51975} x^2 - \frac{31340489}{34054020} x^3 \right] + \ln\left(\frac{m_2^2}{\mu^2}\right) \frac{324148}{19845} + \ln^2\left(\frac{m_2^2}{\mu^2}\right) \ln\left(\frac{m_1^2}{\mu^2}\right) \frac{156992}{6615} \\
& + \ln\left(\frac{m_2^2}{\mu^2}\right) \ln\left(\frac{m_1^2}{\mu^2}\right) \left[\frac{128234}{3969} - \frac{112669}{330750} x + \frac{98746}{51975} x^2 + \frac{31340489}{17027010} x^3 \right] + \ln\left(\frac{m_2^2}{\mu^2}\right) \ln^2\left(\frac{m_1^2}{\mu^2}\right) \frac{68332}{6615} \\
& + \ln\left(\frac{m_1^2}{\mu^2}\right) \left[\frac{83755534727}{583443000} + \frac{78496}{2205} \zeta_2 + \frac{1406143531}{69457500} x + \frac{105157957}{180093375} x^2 - \frac{2287164970759}{7669816654500} x^3 \right] \\
& + \ln^2\left(\frac{m_1^2}{\mu^2}\right) \left[\frac{2668087}{79380} + \frac{112669}{661500} x - \frac{49373}{51975} x^2 - \frac{31340489}{34054020} x^3 \right] + \ln^3\left(\frac{m_1^2}{\mu^2}\right) \frac{412808}{19845} \Big\} \\
& + T_F^2 C_F \left\{ - \frac{3161811182177}{71471767500} + \frac{447392}{19845} \zeta_3 + \frac{9568018}{4862025} \zeta_2 - \frac{64855635472}{2552563125} x + \frac{1048702178522}{97070329125} x^2 + \frac{1980566069882672}{2467763508585375} x^3 \right. \\
& + \ln\left(\frac{m_2^2}{\mu^2}\right) \left[\frac{1786067629}{204205050} - \frac{111848}{15435} \zeta_2 - \frac{128543024}{24310125} x - \frac{22957168}{3361743} x^2 - \frac{2511536080}{2191376187} x^3 \right] \\
& + \ln^2\left(\frac{m_2^2}{\mu^2}\right) \left[\frac{3232799}{4862025} + \frac{752432}{231525} x + \frac{177944}{40425} x^2 + \frac{127858928}{42567525} x^3 \right] - \ln^3\left(\frac{m_2^2}{\mu^2}\right) \frac{111848}{19845} - \ln^2\left(\frac{m_2^2}{\mu^2}\right) \ln\left(\frac{m_1^2}{\mu^2}\right) \frac{223696}{46305} \\
& + \ln\left(\frac{m_2^2}{\mu^2}\right) \ln\left(\frac{m_1^2}{\mu^2}\right) \left[\frac{22238456}{4862025} - \frac{1504864}{231525} x - \frac{355888}{40425} x^2 - \frac{255717856}{42567525} x^3 \right] + \ln\left(\frac{m_2^2}{\mu^2}\right) \ln^2\left(\frac{m_1^2}{\mu^2}\right) \frac{223696}{46305} \\
& + \ln\left(\frac{m_1^2}{\mu^2}\right) \left[\frac{24797875607}{1021025250} - \frac{111848}{15435} \zeta_2 + \frac{128543024}{24310125} x + \frac{22957168}{3361743} x^2 + \frac{2511536080}{2191376187} x^3 \right] \\
& \left. + \ln^2\left(\frac{m_1^2}{\mu^2}\right) \left[\frac{3232799}{4862025} + \frac{752432}{231525} x + \frac{177944}{40425} x^2 + \frac{127858928}{42567525} x^3 \right] - \ln^3\left(\frac{m_1^2}{\mu^2}\right) \frac{1230328}{138915} \right\} + O(x^4 \ln^3(x))
\end{aligned}$$

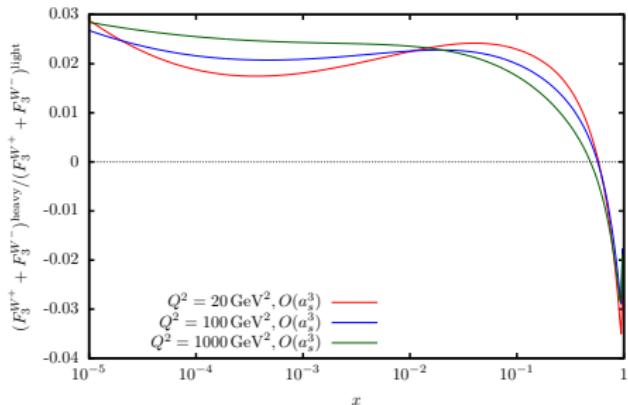
→ q2e/exp [Harlander, Seidensticker, Steinhauser 1999] $x = m_1^2/m_2^2$

Analytic general N results are available for $A_{qq,Q}^{\text{NS}}$, A_{Qq}^{PS} and the scalar integrals of $A_{gg,Q}$.

NS corrections to $g_{1(2)}(x, Q^2)$ and $xF_3^{W^+ + W^-}$



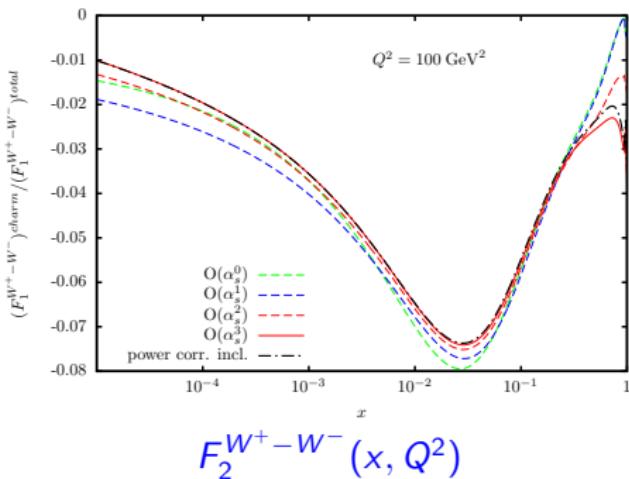
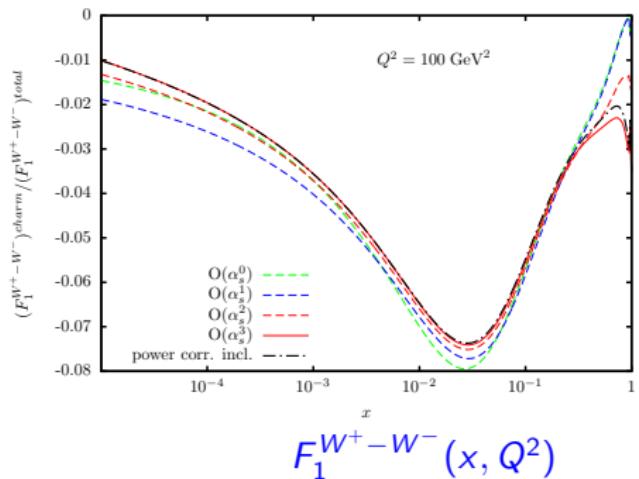
$$g_1(x, Q^2)$$



$$xF_3^{W^+ + W^-}(x, Q^2)$$

The corrections to $g_2(x, Q^2)$ are obtained using the Wandzura-Wilczek relation.

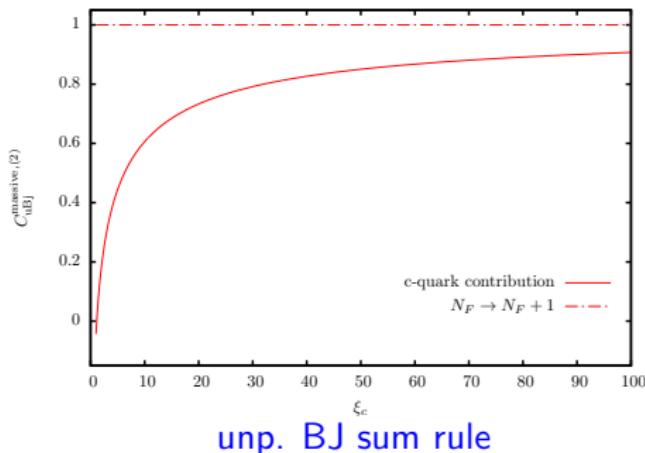
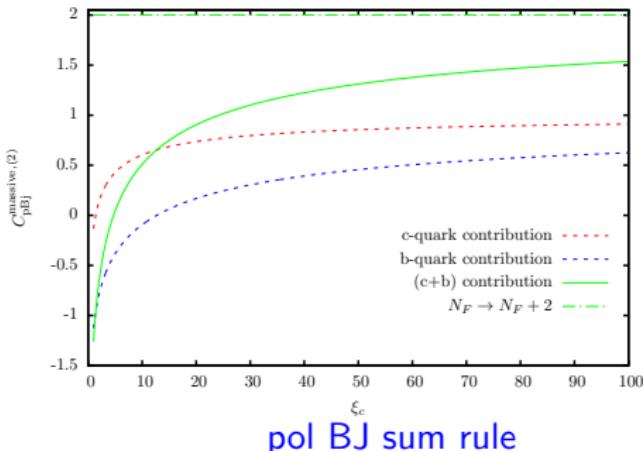
NS corrections to $F_1^{W^+ - W^-}$ and $F_2^{W^+ - W^-}$



The massless corrections are due to Davies, Vogt, Moch, Vermaseren,
LT-1084.
(from: A. Behring et al., DESY 16-148)

Sum Rules and Integral Relations

$O(\alpha_s^2)$ Complete NS corrections



Note the negative corrections at low Q^2 !

Already for charm it takes quite a while to become massless.

JB, G. Falcioni, A. De Freitas, Nucl. Phys. B910 (2016) 568.

Related results: Gross-Llewellyn Smith relation.

Integral Relations in the Polarized Case

JB, Kochlev 1996; JB, Tkabladze 1998 (including the weak sector: NC and CC).

$$g_2^{\tau=2}(x, Q^2) = -g_1^{\tau=2}(x, Q^2) + \int_x^1 \frac{dy}{y} g_1^{\tau=2}(y, Q^2) \quad \text{WW-relation}$$

$$g_3^{\tau=2}(x, Q^2) = 2x \int_x^1 \frac{dy}{y} g_4^{\tau=2}(y, Q^2)$$

$$g_1^{\tau=3}(x, Q^2) = \frac{4M^2 x^2}{Q^2} \left[g_2^{\tau=3}(x, Q^2) - 2 \int_x^1 \frac{dy}{y} g_2^{\tau=3}(y, Q^2) \right]$$

$$\frac{4M^2 x^2}{Q^2} g_3^{\tau=3}(x, Q^2) = g_4^{\tau=3}(x, Q^2) \left(1 + \frac{4M^2 x^2}{Q^2} \right) + 3 \int_x^1 \frac{dy}{y} g_4^{\tau=3}(y, Q^2)$$

$$2x g_5^{\tau=3}(x, Q^2) = - \int_x^1 \frac{dy}{y} g_4^{\tau=3}(y, Q^2)$$

Conclusions

- ▶ 2009: 10-14 Mellin Moments for all massive 3-loop OMEs, WC.
- ▶ 2010: Wilson Coefficients $L_q^{(3),\text{PS}}(N)$, $L_g^{(3),\text{S}}(N)$.
- ▶ 2013: Ladder, V-Graph and Benz-topologies for graphs, with no singularities in ε can be systematically calculated for **general N** .
- ▶ Here **new functions** occur (including a larger number of root-letters in iterated integrals).
- ▶ 2014 $L_q^{\text{NS},(3)}$, $A_{gq,Q}^{\text{S},(3)}$, $A_{qq,Q}^{\text{NS,TR}(3)}$, $H_{2,q}^{\text{PS}(3)}$ and $A_{Qq}^{\text{PS}(3)}$ were completed.
- ▶ A method for the calculation of **graphs with two massive lines** of equal masses and operator insertions has been developed and applied $A_{gg,Q}^{(3)}$.
- ▶ The method can be generalized to the case of unequal masses. Here the moments for $N = 2, 4, 6$ for all graphs with two quark lines of unequal masses are now known [\rightarrow extended renormalization]; for some OMEs the complete 2-mass structure has been computed.
- ▶ The $O(\alpha_s^2)$ charged current Wilson coefficients have been completed.

Conclusions

- ▶ The corresponding 3-loop anomalous dimensions were computed, those for **transversity** for the first time ab initio; those for the **PS-case** independently for the first time.
- ▶ In all NS-cases [NC and CC] we also computed **all power corrections at $O(a_s^2)$** and the associated sum rules in the inclusive case improving an earlier result by JB & W. van Neerven.
- ▶ All master integrals based on iterative integrals over **whatsoever alphabet** for $A_{gg,Q}^{(3)}$ and $A_{Qg}^{(3)}$ have been computed and $A_{gg,Q}^{(3)}$ is known for any even integer moment $N \geq 2$. Here all the topologies, including the ladder- and V-topologies have been solved.
- ▶ We have all the principal means to reconstruct $A_{Qg}^{(3)}$ systematically at very high accuracy. The full analytic solution will request more mathematical efforts.
- ▶ Different new computer-algebra and mathematical technologies were developed. These efforts will continue. The technologies are certainly useful for various present and upcoming calculations for the LHC and ILC.
- ▶ **The EIC can greatly contribute to the precision program for PDFs α_s , m_c , and to unravel twist 3 and more globally detail higher twist.**

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